Expectation and variance of discrete random variables

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EXAMPLE

Imagine you work at a Norwegian bank that issues small business loans of 100,000 NOK.

From historical data, you know that:

- **90%** of customers **repay** the loan in full, plus 20,000 NOK in interest, so the bank **earns 20,000 NOK** profit per loan.
- **10%** of customers **default**, meaning the bank **loses** the full **100,000 NOK** loan.

Is this a good business model?

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If we issue 100 of these loans, some customers will likely repay, while others will likely default.

We can use the **probability distribution** to estimate our potential earnings and losses.

x	20 000	-100 000
$p(x)$	0.90	0.10

We can approximate how many of the 100 customers will repay by multiplying the probability of repaying (**0.90**) by **100.**

Since we earn **20 000** each time a loan is repaid, we can estimate our total earnings by multiplying the expected number of customers who repay by **20 000**

$$(0.90 \cdot 100) \cdot 20\ 000$$

← The entire term represents how much money we expect to earn from 100 customers

↑ $(0.90 \cdot 100)$: Expected number of customers who repay

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x	20 000	-100 000
$p(x)$	0.90	0.10

We can approximate how many of the 100 customers will default by multiplying the probability of defaulting (**0.10**) by **100.**

Since we lose **100 000** each time a loan is defaulted, we can estimate our total loss by multiplying the expected number of customers who defaults by **100 000**

$$(0.90 \cdot 100) \cdot 20\,000 \quad (0.10 \cdot 100) \cdot (-100\,000)$$

Expected number of customers who defaults

The entire term represents how much money we expect to lose from 100 customers

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We can add the two terms together to determine our **expected profit** from the 100 customers

$$(0.90 \cdot 100) \cdot 20\,000 \;+\; (0.10 \cdot 100) \cdot (-100\,000) \;=\; 800\,000$$

↑

Expected profit from 100 customers

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	x		20 000	-100 000
	$p(x)$		0.90	0.10

We can also compute the **average expected profit** per customer by dividing the total profit by 100.

$$\frac{(0.90 \cdot 100) \cdot 20\,000 + (0.10 \cdot 100) \cdot (-100\,000)}{100} = \frac{800\,000}{100} = 8000$$

Expected average profit

$$E(X) = \square\ \square\quad \square\ \square = \sum_{\text{all } x} x \cdot p(x)$$

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Definition

The **expected value** of a random variable X is a weighted average of the possible values of X, where the weights are the probabilities of those values.

$$\mu_X = E(X) = \sum_{all\ x} x \cdot p(x)$$

- The average value of X in the long run

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Example: Rolling a pair of dice. X = Sum

Value of X	2	3	4	5	6	7	8	9	10	11	12
Probability	0.0278	0.0556	0.0833	0.1111	0.1389	0.1667	0.1389	0.1111	0.0833	0.0556	0.0278

$$E(X) = 2 \cdot 0.0278 + 3 \cdot 0.0556 + 4 \cdot 0.0833 + \cdots + 12 \cdot 0.0278 = 7$$

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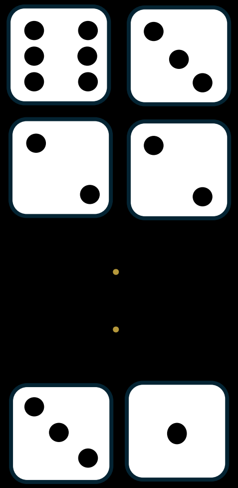
The long-run average of X will approach the expectation.



$$X_1 = 9$$

$$X_2 = 4$$

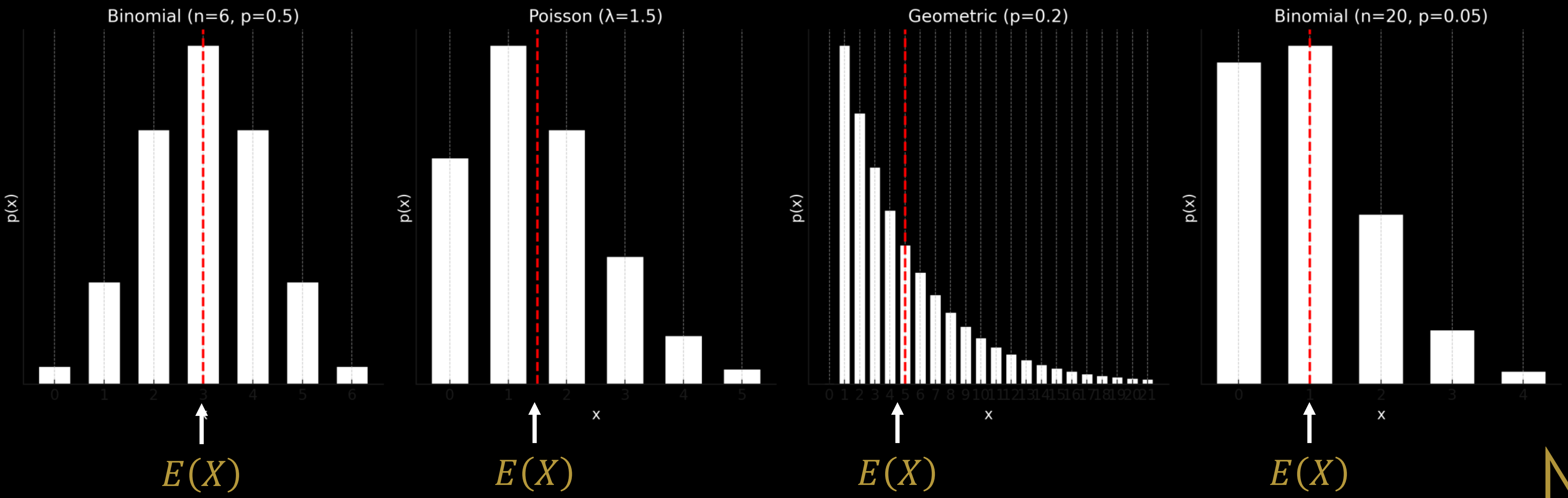
$$\vdots$$

$$X_{1000000} = 4$$

$$\bar{X} = \frac{1}{1000000} \sum X_i = 7.00001$$

This result is called «The law of large numbers»

Measure of center



VARIANCE

The **variance** of X is the expectation of the squared distance between X and its mean:

$$\sigma_X^2 = \mathrm{Var(X)} = \mathrm{E(X} - \mu_X)^2 = \sum_{all\ x} (x - \mu_x)^2 \cdot p(x)$$

- The variance of X in the long run
- A measure of spread in the probability distribution

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Variance – a shortcut formula

The variance of X can be computed with the formula:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= E(X^2) - \mu_X^2$$

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X = Sum of two dice. The probability distribution of X is given by:

Value of X	2	3	4	5	6	7	8	9	10	11	12
Probability	0.0278	0.0556	0.0833	0.1111	0.1389	0.1667	0.1389	0.1111	0.0833	0.0556	0.0278

$$\mu_x = E(X) = 2 \cdot 0.0278 + 3 \cdot 0.0556 + 4 \cdot 0.0833 + \cdots + 12 \cdot 0.0278 = 7$$

$$E(X^2) = 2^2 \cdot 0.0278 + 3^2 \cdot 0.0556 + 4^2 \cdot 0.0833 + \cdots + 12^2 \cdot 0.0278 = 54.83$$

$$Var(X) = E(X^2) - \mu_x^2 = 54.83 - 7^2 = 5.83$$

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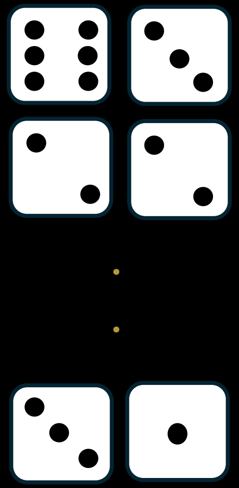

$$X_1 = 9$$

$$X_2 = 4$$

$$\vdots$$

$$X_{1000000} = 4$$

$$s^2 = \frac{1}{1000000 - 1} \sum (X_i - \bar{X})^2 = 5.83001$$

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