Maximum Likelihood Estimation

Bernoulli distribution

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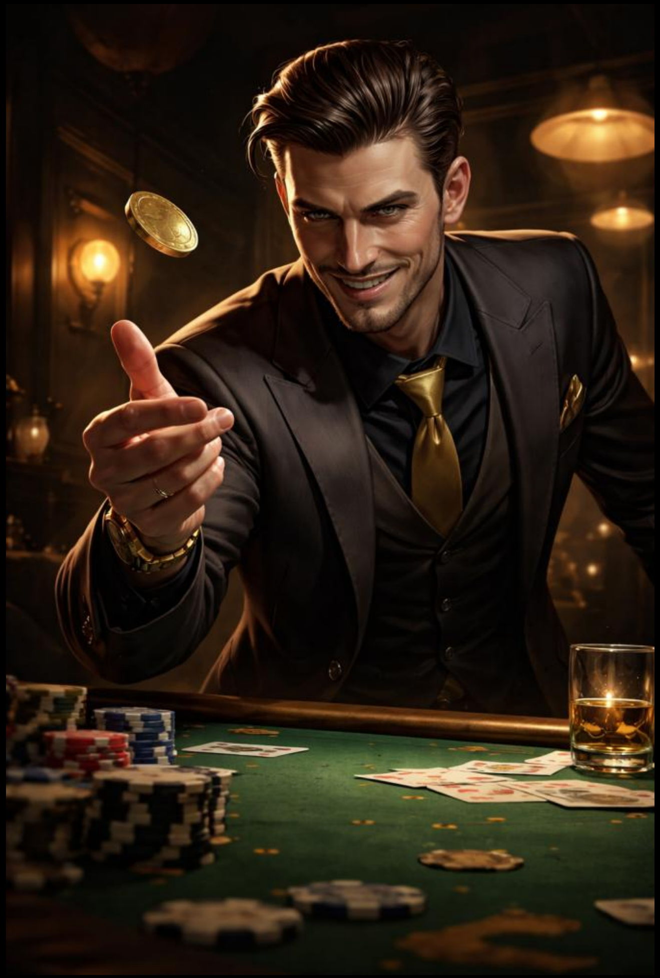
Bernoulli experiment

- Is the coin fair?
- Flip the coin many times and calculate the relative frequency of heads
  - Close to 0.5 ✅
  - Far from 0.5 ❌
- A maximum likelihood estimate



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Step 1: Assume a distribution

$$X_1, X_2, \dots, X_n \sim \text{Benoulli}(p)$$

$$P(X_i = x) = p^x(1-p)^{1-x}, \qquad x = 0,1.$$

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Step 2: Find the likelihood function

(expression for the joint distribution as a function of p)

$$P(X_i = x) = p^x(1-p)^{1-x}, \qquad x = 0{,}1.$$

$$L(p) = f(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} P(X_i = x_i) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$L(p) = p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i}$$

$$\prod_{i=1}^{n} p^{x_i} \qquad = p^{x_1}p^{x_2}\cdots p^{x_n} = p^{\sum x_i}$$

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Step 2: Find the log-likelihood function

$$L(p) = p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i}$$

$$\ell(p) = \log L(p) = \log(p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i})$$

$$= \log p \sum_{i=1}^{n} x_i + \log(1-p)\left(n - \sum_{i=1}^{n} x_i\right)$$

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Step 3: Maximize the log-likelihood function

$$\ell(p) = \sum_{i=1}^{n} x_i \log(p) + (n - \sum_{i=1}^{n} x_i) \log(1-p)$$

$$\frac{\partial \ell}{\partial p} = \frac{\sum x_i}{p} + \frac{n - \sum x_i}{1-p}(-1) = 0$$

$$\frac{\sum x_i}{p} = \frac{n - \sum x_i}{1-p} \qquad \cdot\, p(1-p)$$

$$(1-p)\sum x_i = p(n - \sum x_i)$$

$$\sum x_i - \cancel{p\sum x_i} = np - \cancel{p\sum x_i}$$

$$\sum x_i = np$$

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Step 3: Maximize the log-likelihood function

$$\sum x_i = np$$

$$\hat{p} = \frac{\sum x_i}{n} = \bar{x}$$

The maximum likelihood estimator is just the relative frequency of heads!

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