Monte Carlo Simulation

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What is monte Carlo simulation?

- Computational technique
- Model and analyse systems influenced by randomness
- Repeated random sampling
- Law of large numbers
- Very useful for casino games (hence the name)

Let X be a random variable with distribution f.

It is hard to find an explicit expression for the distribution f, but we can simulate from it.

Say we are interested in finding $E(g(X))$

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Simple Monte Carlo Simulation

- Simulate $X_1, X_2, \ldots, X_n$ by drawing from f (e.g. n = 10 000)
- Approximate E(g(X)) by law of large numbers:

$$E\left(g(X)\right) \approx \frac{1}{10\,000} \sum_{i=1}^{10\,000} g(X_i)$$

If you are interested in $P(X \geq x)$, let $g(X) = I(X \geq x)$.

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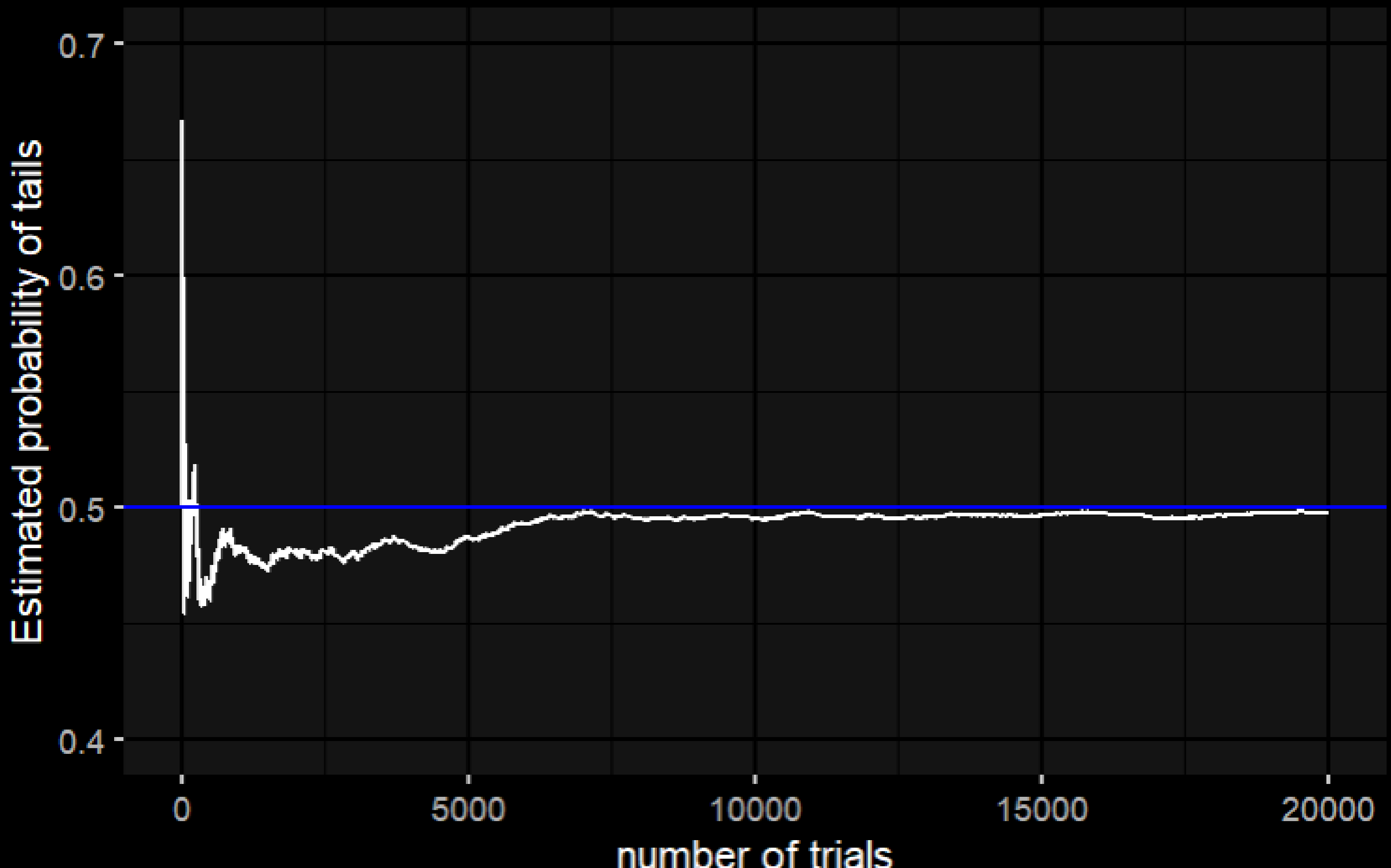
Estimated probability of tails
0.7
0.6
0.5
0.4
0
5000
10000
15000
20000
number of trials

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Four step Recipe

1. Define a domain of possible values
2. Generate random numbers within that domain from a probability distribution
3. Perform a computation using the random numbers
4. Combine the results across many repetitions

Create a simulation model of the process

Simulate the process many times

Combine the results using law of large numbers

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Example: Yatzy

- Roll five dice three times
- After each roll, take out the dice you want to keep
- In each round, you have a specific outcome you want to achieve, e.g. maximum number of 2’s.

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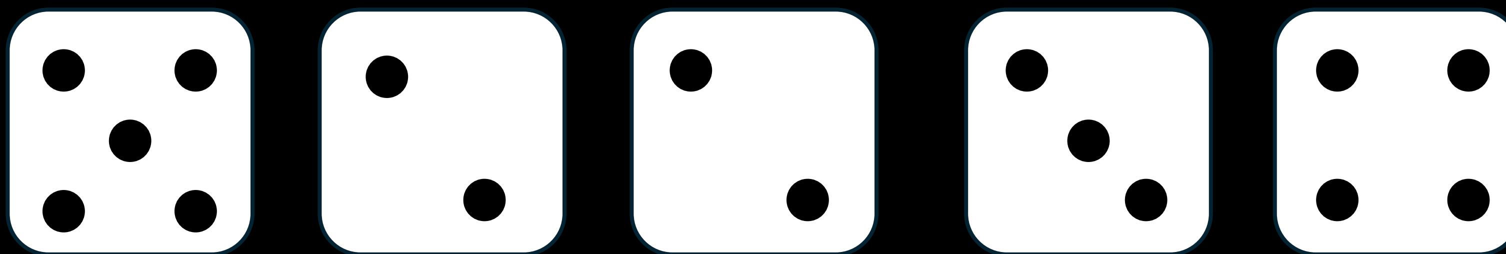

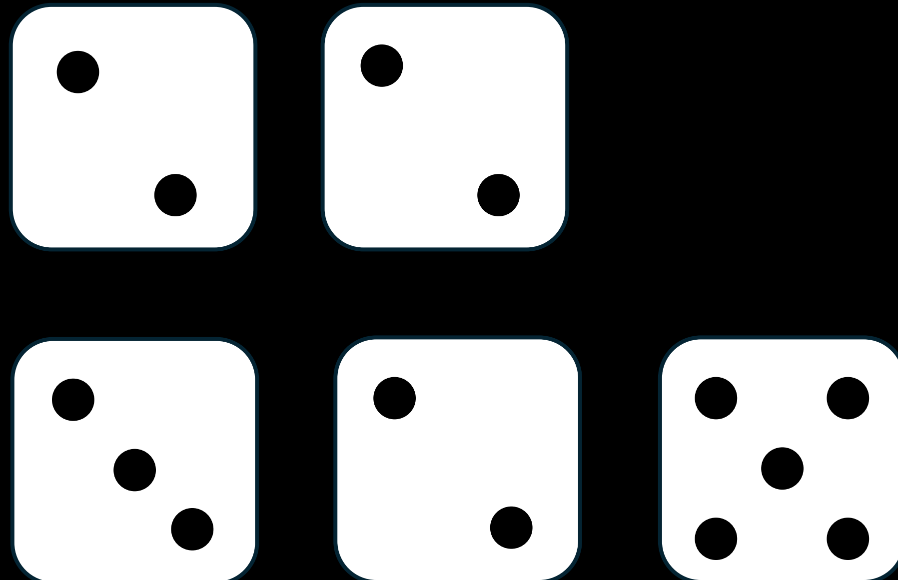

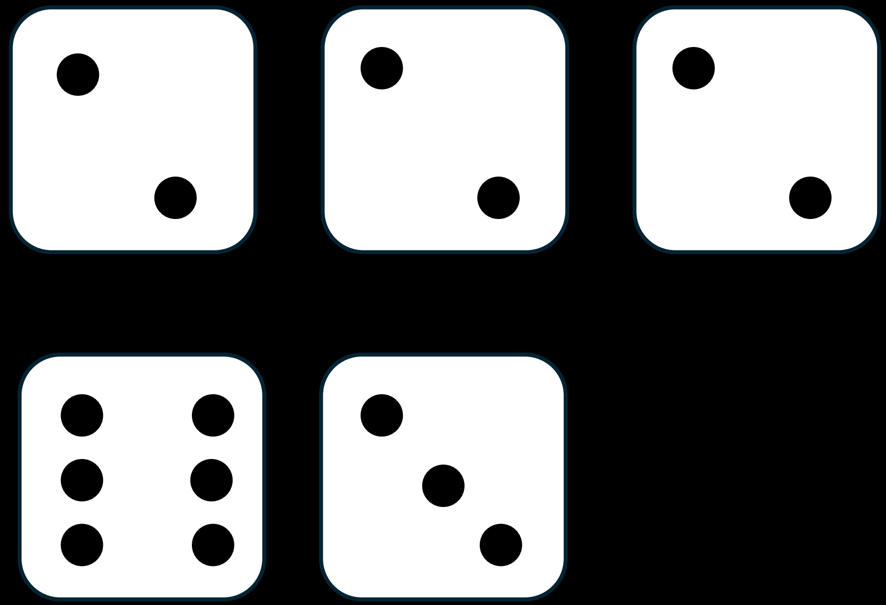

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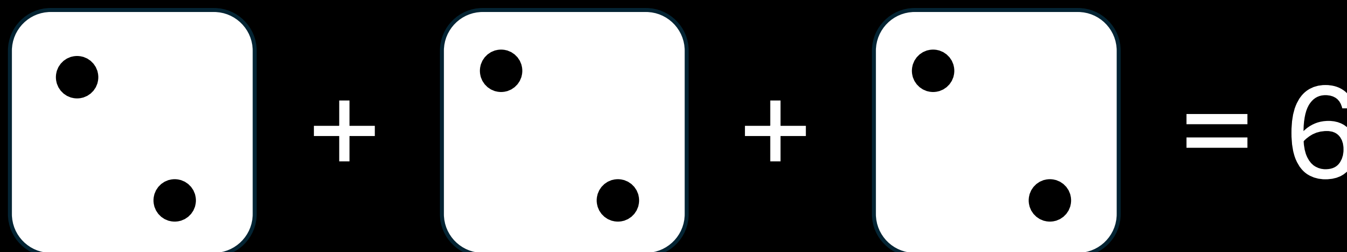
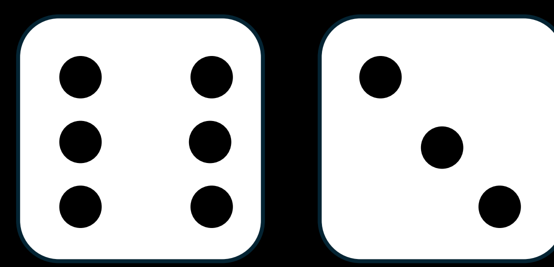
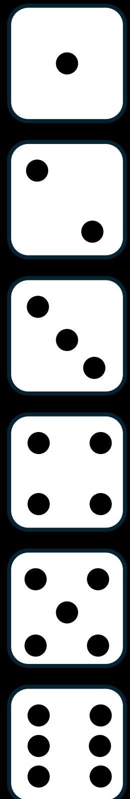

+
+
= 6





	4
	6
	6
	12
	5
	12
Total	**45**
Bonus	+50 if Total ≥ 63

1. What is the expected total score?
2. What is the probability of getting the bonus?
3. What does the total score distribution look like?

X_{ji} denote the number of new dices showing i dots after j throws in round i

Y_i: The score of round number i

$Y = \sum_{i=1}^{6} Y_i$: Total score.

$$Y_i = (X_{1i} + X_{2i} + X_{3i}) \cdot i$$

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Then

$$X_{1i} \sim Bin\left(n = 5, p = \frac{1}{6}\right)$$

$$X_{2i}|X_{1i} \sim Bin\left(5 - X_{1i}, \frac{1}{6}\right)$$

$$X_{3i}|X_{1i}, X_{2i} \sim Bin\left(5 - X_{1i} - X_{2i}, \frac{1}{6}\right)$$

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Assume i = 1 and drop the i index for now.
We have

$$E(X_1) = \frac{5}{6},$$

$$E(X_2 | X_1) = \frac{5 - X_1}{6},$$

$$E(X_3 | X_1, X_2) = \frac{5 - X_1 - X_2}{6}$$

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$$
\begin{aligned}
E(Y_1) &= E(X_1 + X_2 + X_3) \\
&= EX_1 + E(EX_2|X_1) + E(EX_3|X_2, X_1) \\
&= \frac{5}{6} + E\frac{5 - X_1}{6} + E\frac{5 - X_1 - X_2}{6} \\
&= \frac{5}{6} + \frac{5 - 5/6}{6} + \frac{5 - 5/6 - EE(X_2|X_1)}{6} \\
&= \frac{5}{6} + \frac{5 - 5/6}{6} + \frac{5 - 5/6 - E\frac{5 - X_1}{6}}{6} \\
&= \frac{5}{6} + \frac{5 - 5/6}{6} + \frac{5 - 5/6 - 5/6 + 5/36}{6} \\
&= \frac{30}{36} + \frac{25}{36} + \frac{180 - 60 + 5}{216} \\
&= \frac{180 + 150 + 125}{216} = \frac{455}{216} \approx 2.1
\end{aligned}
$$

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$$E(Y) = \sum_{i=1}^{6} E(Y_j) = E(Y_1) \sum_{i=1}^{6} j$$

$$= \frac{455}{211}(1 + 2 + 3 + 4 + 5 + 6)$$

$$= \frac{455}{211} \cdot 21 \approx 44.24$$

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Monte carlo simulation: Yatzy

For round $i \in \{1, 2, 3, 4, 5, 6\}$

Simulate rolling 5 dices

$$x_{1i} \sim Bin(5, \frac{1}{6})$$

Simulate rolling $5 - x_{1i}$ dices:

$$x_{2i} \sim Bin(5 - x_{1i}, \frac{1}{6})$$

Simulate rolling $5 - x_{1i} - x_{2i}$ dices:

$$x_{3i} \sim Bin(5 - x_{1i} - x_{2i}, \frac{1}{6})$$

$$y_i = (x_{1i} + x_{2i} + x_{3i}) \cdot i$$

$$y = \sum_{i=1}^{6} y_i$$

Simulation model

Repeat the game 10 000 times

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What is the expected total score?

$$44.24 = E(Y) \approx \frac{1}{10\,000} \sum_{b=1}^{10\,000} y^b = 44.25$$

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What is the probability of getting the bonus?

$$P(Y \geq 63) \approx \frac{1}{10\,000} \sum_{b=1}^{10\,000} I\left(y^b \geq 63\right) = 0.0426 = 4.26\%$$

$$\mathrm{I}(y^b \geq 63) = \begin{cases} 1, & if\ y^b \geq 63 \\ 0, & otherwise \end{cases}$$

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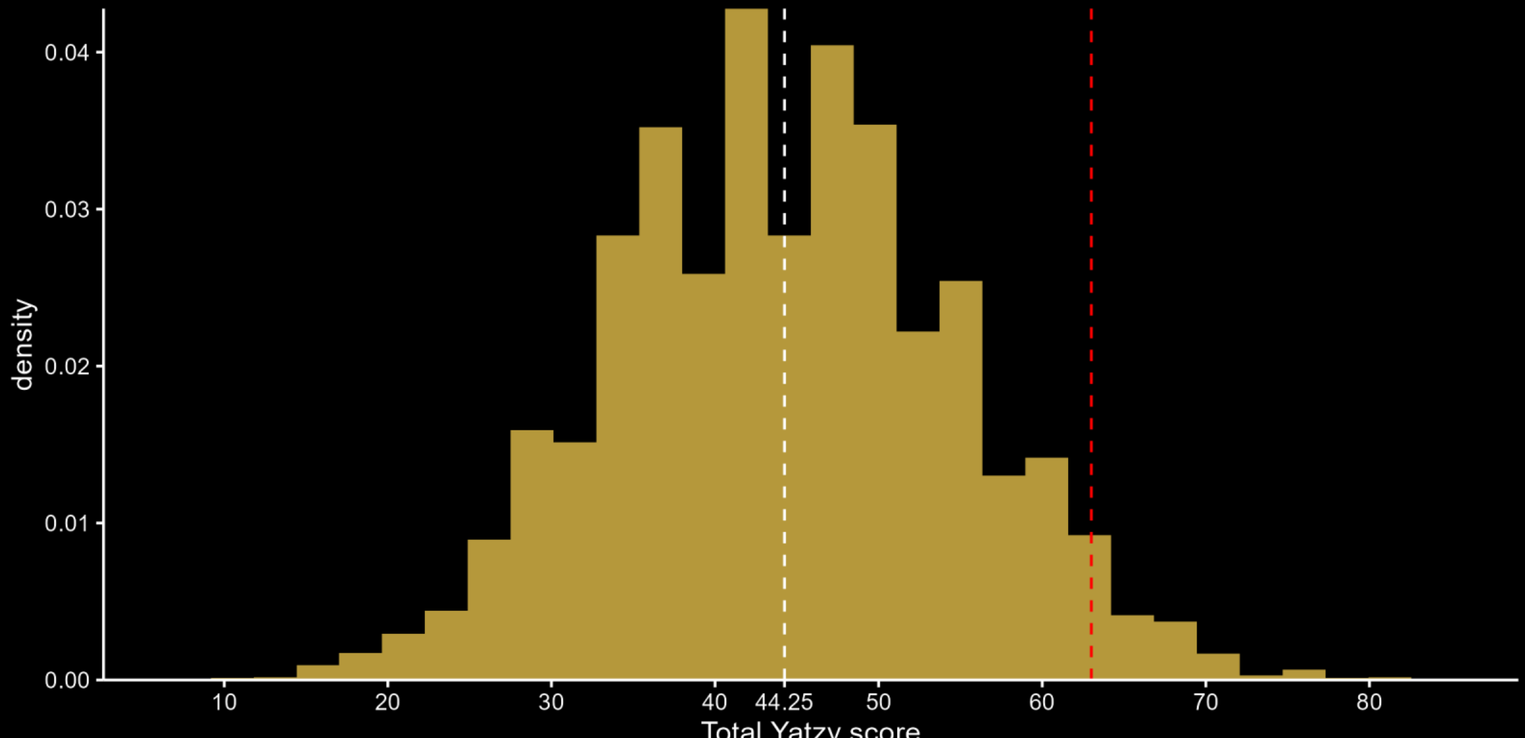
How does the score distribution look?



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